Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2 + 1) Dimensions

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We explicitly derive the duality between a free electronic Dirac cone and quantum electrodynamics in (2 + 1) dimensions (QED_3) with $N = 1$ fermion flavors. The duality proceeds via an exact, nonlocal mapping from electrons to dual fermions with long-range interactions encoded by an emergent gauge field. This mapping allows us to construct parent Hamiltonians for exotic topological-insulator surface phases, derive the particle-hole-symmetric field theory of a half-filled Landau level, and nontrivially constrain QED_3 scaling dimensions. We similarly establish duality between bosonic topological insulator surfaces and $N = 2$ QED_3.

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Introduction.—Recent theoretical work has revealed intricate connections between the following rather different physical systems: metallic surfaces of three-dimensional topological insulators (TIs) [1–5]; composite Fermi liquids (CFLs) [6–12] that arise in a strong magnetic field when the lowest Landau level is half filled; and quantum electrodynamics in (2 + 1) dimensions (QED_3). Central to these surprising relationships is a newly discovered duality between a single Dirac cone with action

$$\mathcal{S}_{\rm Dirac} = \int_{t,x,y} i\bar{\Psi}\gamma^\mu(\partial_\mu - iA_\mu)\Psi, \qquad (1)$$

and QED_3 with $N = 1$ fermion flavors,

$$\mathcal{S}_{\rm QED_3} = \int_{t,x,y}\left[i\bar{\tilde{\Psi}}\gamma^\mu(\partial_\mu - ia_\mu)\tilde{\Psi} + \frac{\epsilon_{\mu\nu\kappa}A_\mu\partial_\nu a_\kappa}{4\pi} + \cdots\right]. \qquad (2)$$

Here, γ^μ are 2×2 Dirac matrices (with $\mu = 0$ being the temporal component and $\mu = 1$, 2 spatial components), the ellipsis denotes additional allowed contributions such as the Maxwell term $\sim(\epsilon_{\mu\nu\kappa}\partial_\nu a_\kappa)^2$, and A_μ is the electromagnetic vector potential. The two-component spinor Ψ thus carries electric charge while $\tilde{\Psi}$ is neutral; in the QED_3 formulation electric currents are instead encoded through fluxes of the dynamical gauge field a_μ.

Equation (1) captures two setups of interest here. (i) Without an applied magnetic field ($B \equiv \partial_1 A_2 - \partial_2 A_1 = 0$), the theory describes the time-reversal-preserving Dirac cone routinely observed in TI surfaces. In this context Eq. (2) provides an equivalent surface description in which the emergence of certain strongly interacting symmetric gapped phases [13–16] becomes extremely natural [17,18]. (ii) Alternatively, one can view Eq. (1) as a Dirac theory enjoying an exact particle-hole symmetry [18,19] that pins the chemical potential to the Dirac point ($A_0 = 0$) while permitting a finite magnetic field ($B \neq 0$) [20]. The field rearranges the spectrum into Landau levels symmetric about zero energy. In particular, one level sits exactly at zero energy and is constrained to half filling by particle-hole symmetry. At low energies the system maps precisely onto the half-filled lowest Landau level in a conventional two-dimensional electron gas; interactions can thus generate a CFL. For this high-field problem Eq. (2) describes the particle-hole-symmetric CFL reformulation introduced recently by Son [21], with $\tilde{\Psi}$ interpreted as the appropriate composite fermion field; see also [19,22–24].

Two methods have been employed to support the duality between Eqs. (1) and (2) [17–19,25]. First, various known TI surface phases were shown to be accessible in either formulation; second, an electric-magnetic duality for a (gauged) TI bulk was shown to recover the alternate surface theory in Eq. (2). We develop a new operator-based derivation of this duality directly in two spatial dimensions by relating the fermions Ψ and $\tilde{\Psi}$ and explicitly mapping the path integrals onto one another. As we will see, the emergent gauge field a_μ reflects the nonlocal relation between the two fermion fields. Our essentially exact mapping allows us to demonstrate that QED_3 regularized as in our construction shares the same low-energy behavior as a free Dirac cone, leading to nontrivial predictions for operator scaling dimensions in QED_3. We bolster this conclusion with several consistency checks [26]. Furthermore, the mapping provides a derivation of Son's proposed CFL field theory [21]. That is, the half-filled Landau level problem that occurs in $\mathcal{S}_{\rm Dirac}$ for $A_0 = 0$, $B \neq 0$ maps to dual fermions $\tilde{\Psi}$ coupled to a dynamical gauge field without a Chern-Simons term, and where the dual fermions are doped to a density matching the original magnetic flux. Generalizing the method to bosonic TI surfaces [33] allows us to obtain a dual description given by QED_3 with $N = 2$ fermion flavors.

Model and symmetries.—Our analysis begins from an array of one-dimensional chiral-electron wires, sketched in Fig. 1, with Hamiltonian $H = \int_x \sum_y (h_{\rm wire} + h_{\rm hop})$ given by

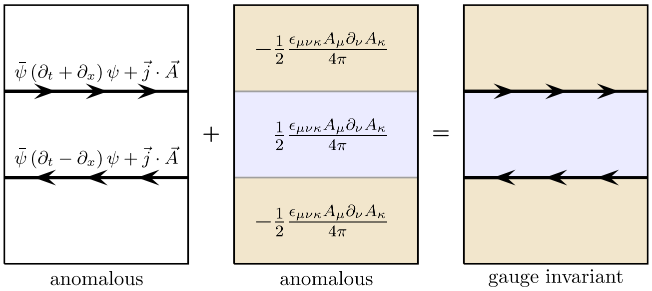

FIG. 1. An array of chiral wires (left) exhibits a gauge anomaly when coupled to the electromagnetic vector potential A_μ. The same anomaly but with opposite sign arises in a two-dimensional Chern-Simons theory with alternating coefficient (middle). The anomalies exactly cancel in the full two-dimensional theory (right) containing both the chiral wires and the bulk Chern-Simons theory.

$$h_{\text{wire}} = v(-1)^y \psi_y^\dagger(-i\partial_x)\psi_y, \tag{3}$$

$$h_{\text{hop}} = -w(-1)^y(\psi_y^\dagger \psi_{y+1} + \text{H.c.}). \tag{4}$$

Here, $\psi_y^\dagger(x)$ creates a chiral electron at coordinate x in wire y; electrons move rightward with speed v for even y and leftward for odd y. Equation (3) describes the intrawire kinetic energy while Eq. (4) encodes interwire hopping with strength w. (Coupling to the vector potential A_μ will be discussed shortly.) The Hamiltonian manifestly conserves charge and is invariant under the antiunitary symmetries $\mathcal{T}$, $\mathcal{C}$ defined as

$$\mathcal{T}\psi_y\mathcal{T}^{-1} = (-1)^y\psi_{y+1}, \qquad \mathcal{C}\psi_y\mathcal{C}^{-1} = \psi_{y+1}^\dagger. \tag{5}$$

The band structure derived from Eqs. (3) and (4) supports a single Dirac cone centered about zero momentum. Upon defining a slowly varying fermionic spinor $\Psi(x,y) \equiv [\psi_{2y}(x), \psi_{2y+1}(x)]^T$ one can readily take the continuum limit, which yields an effective Dirac Hamiltonian $H_{\text{eff}} = \int_{x,y}\Psi^\dagger[v\sigma^z(-i\partial_x) + w\sigma^y(-i\partial_y)]\Psi$ corresponding to Eq. (1) ($\sigma^{x,y,z}$ are Pauli matrices; the associated Dirac matrices are $\gamma^0 = \sigma^x, \gamma^1 = -i\sigma^y, \gamma^2 = i\sigma^z$). Moreover, symmetries act on the continuum fields according to

$$\mathcal{T}\Psi\mathcal{T}^{-1} = i\sigma^y\Psi, \qquad \mathcal{C}\Psi\mathcal{C}^{-1} = \sigma^x\Psi^\dagger. \tag{6}$$

Viewed from the low-energy limit, $\mathcal{T}$ reproduces the time-reversal transformation familiar from the TI surface while $\mathcal{C}$ is the particle-hole symmetry relevant for the half-filled Landau level setting [26]. Our wire model therefore captures both systems of interest—but in a way that facilitates an explicit duality transformation. Note that the continuum and wire models, H_{eff} and H, are both anomalous in two dimensions (with $\mathcal{T}$ or $\mathcal{C}$) but may exist on the surface of three-dimensional TIs (see also Ref. [26]).

When coupling the wire model Eqs. (3) and (4) to the external electromagnetic vector potential A_μ, some care must be taken to ensure gauge invariance. Each chiral-electron wire is on its own anomalous and cannot be made gauge invariant in a purely $(1+1)$-dimensional system. This situation is familiar from the chiral edge state of a quantum Hall system. There, gauge invariance is maintained due to the bulk Chern-Simons term for the vector potential [34]. For the present case, an alternating Chern-Simons term for A_μ between wires,

$$\mathcal{S}_{\text{staggered CS}} = -\sum_y \frac{1}{2}\frac{(-1)^y}{4\pi}\int_{t,x}\int_y^{y+1} dy' \epsilon_{\mu\nu\kappa}A_\mu\partial_\nu A_\kappa, \tag{7}$$

ensures gauge invariance as Fig. 1 sketches. See Supplemental Material [26] for details in the wire model. The fractional Chern-Simons coefficient can be understood, e.g., if one views the wire model as describing a $\mathcal{T}$-symmetric "antiferromagnetic" TI [35–37] surface decorated with alternating magnetic strips that generate a staggered Hall conductance $\sigma_{xy} = \pm e^2/(2h)$. In the continuum limit this oscillating term drops out and we obtain $\mathcal{S}_{\text{Dirac}}$ in Eq. (1).

Dual fermions.—To perform the duality we adopt a bosonized description, writing

$$\psi_y(x) = \eta_y e^{i\phi_y(x)} \quad \text{(electron)}, \tag{8}$$

where $\phi_y(x)$ are chiral boson fields satisfying $[\phi_y(x), \phi_{y'}(x')] = \delta_{yy'}(-1)^y i\pi\,\text{sgn}(x-x')$ and η_y are Klein factors ensuring anticommutation between fermions in different wires. We define dual fields via

$$\tilde{\phi}_y \equiv \sum_{y'\neq y}\text{sgn}(y-y')(-1)^{y'}\phi_{y'} \equiv \sum_{y'} D_{y,y'}\phi_{y'}. \tag{9}$$

On the right-hand side we introduced matrix notation $D_{y,y'} = (1-\delta_{yy'})\text{sgn}(y-y')(-1)^{y'}$ for later convenience. The dual fields obey

$$[\tilde{\phi}_y(x), \tilde{\phi}_{y'}(x')] = -[\phi_y(x), \phi_{y'}(x')], \tag{10}$$

i.e., they are also chiral bosons with chirality opposite to ϕ_y. We can therefore define a new dual-fermion operator as

$$\tilde{\psi}_y = \eta_y(-1)^{y(y+1)/2}e^{i\tilde{\phi}_y} \quad \text{(dual fermion)}, \tag{11}$$

using the same Klein factors from Eq. (8). (The y-dependent phase inserted above merely simplifies expressions that follow.) This dual fermion is a key result of this paper and closely relates to the familiar composite fermion obtained via Chern-Simons flux attachment [12,38]. Indeed, one may write

$$\tilde{\psi}_y = \begin{cases} \psi_y e^{i(\pi/2)y(y+1)} e^{i\int_{x'}\sum_{y'}\alpha_{y-y'}(x-x')\rho_{y'}(x')}, & \text{for } y \text{ even} \\ \psi_y^\dagger e^{i(\pi/2)y(y+1)} e^{i\int_{x'}\sum_{y'}\alpha_{y-y'}(x-x')\rho_{y'}(x')}, & \text{for } y \text{ odd}, \end{cases} \tag{12}$$

where $\alpha_y(x) = 2\pi\,\text{sgn}(y+0^+)\Theta(-x)$ winds by 4π around the origin (Θ is the Heaviside function). The exponential thus attaches 4π flux (a $2hc/e$ vortex) to electrons or holes. In a crucial difference to conventional flux attachment, however, dual fermions are expressed in terms of chiral fields. This is reflected in $\psi_y^\dagger\psi_y \neq \tilde{\psi}_y^\dagger\tilde{\psi}_y$, unlike usual composite fermions whose density exactly matches the electron density [12].

Dual fermions exhibit the following properties: (1) Applying duality to $\tilde{\psi}_y$ recovers the original electrons [i.e., $D^2 = 1$ for the matrix defined in Eq. (9)]. (2) Upon encircling an electron, the dual fermion acquires a phase of 4π. (3) Dual-fermion hopping between wires is a local process since

$$\tilde{\psi}_y^\dagger\tilde{\psi}_{y+1} = \begin{cases} \psi_{y+1}^\dagger\psi_y, & \text{for } y \text{ even} \\ \psi_y^\dagger\psi_{y+1}, & \text{for } y \text{ odd.} \end{cases} \tag{13}$$

Hence the original electron hopping Hamiltonian in Eq. (4) maps to a dual-fermion hopping Hamiltonian of identical form. Together with the chiral nature of $\tilde{\psi}_y$, this motivates defining slowly varying dual Dirac fermions $\tilde{\Psi}(x,y) = [\tilde{\psi}_{2y}(x), \tilde{\psi}_{2y+1}(x)]^T$ analogous to $\Psi(x,y)$. (4) Time reversal and charge conjugation act locally on the dual Dirac fermions, but their roles are interchanged compared to electrons,

$$\mathcal{T}\tilde{\Psi}\mathcal{T}^{-1} = \sigma^x\tilde{\Psi}^\dagger, \qquad \mathcal{C}\tilde{\Psi}\mathcal{C}^{-1} = i\sigma^y\tilde{\Psi} \tag{14}$$

(up to unimportant overall phase factors); cf. Eq. (6). The local action of $\mathcal{T}$ and $\mathcal{C}$ on the dual fermions is particularly noteworthy since this property is difficult to realize with the standard flux-attachment procedure. (5) Under (unitary) mirror symmetries $\mathcal{M}_x$: $x \to -x$ and $\mathcal{M}_y$: $y \to -y$ dual fermions transform as [26]

$$\mathcal{M}_x\tilde{\Psi}\mathcal{M}_x^{-1} = -\sigma_x\tilde{\Psi}^\dagger, \qquad \mathcal{M}_y\tilde{\Psi}\mathcal{M}_y^{-1} = \tilde{\Psi}^\dagger. \tag{15}$$

Consequently dual currents $\tilde{j}_{x,y} = \tilde{\Psi}^\dagger\sigma^{z,y}\tilde{\Psi}$ are odd under $\mathcal{M}_y$ and $\mathcal{M}_x$, respectively. This is opposite to the transformation of electron currents but expected for vortex currents (vorticity is odd under $\mathcal{M}_x$ and $\mathcal{M}_y$). (6) Density $\tilde{\rho}_y = -(-1)^y/(2\pi)\partial_x\tilde{\phi}_y$ and intrawire kinetic energy $\propto (-1)^y\tilde{\psi}_y^\dagger i\partial_x\tilde{\psi}_y \sim (\partial_x\tilde{\phi}_y)^2$ of dual fermions are nonlocal in terms of electrons and vice versa. This is the origin of an emergent gauge field a_μ in the dual formulation.

Duality of the path integral.—In bosonized language, the (real-time) path integral for the original electrons, assuming $A_\mu = 0$ for now, reads $\mathcal{Z} = \int \mathcal{D}\phi e^{i\mathcal{S}}$ with

$$\mathcal{S} = \int_{t,x}\sum_y\left\{\frac{(-1)^y\partial_x\phi_y\partial_t\phi_y}{4\pi} - h_{\mathrm{wire}}[\phi] - h_{\mathrm{hop}}[\phi]\right\}. \tag{16}$$

As noted above, the interwire hopping term is self dual, i.e., $h_{\mathrm{hop}}[\phi] = h_{\mathrm{hop}}[\tilde{\phi}]$. It further follows from Eq. (10), or equivalently by changing variables in the path integral, that the time-derivative term only acquires a minus sign when mapping $\phi \to \tilde{\phi}$. Intrawire kinetic energy does, however, transform nontrivially under duality,

$$h_{\mathrm{wire}} = \frac{v}{4\pi}(\partial_x\phi_y)^2 = \frac{v}{4\pi}\left(\partial_x\sum_{y'}D_{y,y'}\tilde{\phi}_{y'}\right)^2. \tag{17}$$

Indeed, the expression on the right is highly nonlocal—which is not surprising from the viewpoint of the more familiar boson-vortex dualities, where dual vortices exhibit long-range interactions [39,40].

To obtain a local theory, we formally introduce new integration variables $a_{0,y}(x,t)$ and $a_{1,y}(x,t)$ and add two complete squares to the action by replacing

$$h_{\mathrm{wire}} \to h_{\mathrm{wire}} + \frac{1}{4\pi}\left\{(v+u)(\partial_x\tilde{\phi}_y - a_{1,y})^2 - v\left(2\Delta^{-1,T}[(-1)^y\partial_x\tilde{\phi}_y] + \frac{\Delta[a_{0,y} - v(-1)^y a_{1,y}]}{2v}\right)^2\right\} \tag{18}$$

in Eq. (16). Here, Δ denotes a lattice derivative, e.g., $\Delta a_{\mu,y} = a_{\mu,y+1} - a_{\mu,y}$; $\Delta^{-1,T}$ is its inverse transpose, i.e., $\sum_y \Delta^{-1,T}[a_{\mu,y}]\Delta[a_{\mu,y}] = \sum_y a_{\mu,y}^2$, which defines $\Delta^{-1,T}$ up to a constant. We take $\Delta^{-1,T}a_{\mu,y} = \frac{1}{2}\sum_{y'}\mathrm{sgn}(y'-y+0^+)a_{\mu,y'}$. The shift (18) multiplies the path integral by a benign constant but is nevertheless very useful. Given the expressions for $\Delta^{-1,T}$ and D, one readily verifies that all quadratic terms in $\partial_x\tilde{\phi}_y$ with coefficient v exactly cancel—leaving only local contributions.

The remaining terms can be reorganized to obtain $\mathcal{Z} \sim \int \mathcal{D}\tilde{\phi}\mathcal{D}a_\mu e^{i\mathcal{S}_{\mathrm{dual}}}$ with dual action

$$\mathcal{S}_{\mathrm{dual}} = \int_{t,x}\sum_y\left\{\frac{-(-1)^y\partial_x\tilde{\phi}_y\partial_t\tilde{\phi}_y}{4\pi} - h_{\mathrm{wire}}^{\mathrm{dual}}[\tilde{\phi}, a_\mu] - h_{\mathrm{hop}}[\tilde{\phi}]\right\} + \mathcal{S}_{\mathrm{MW}}[a_\mu] + \mathcal{S}'_{\mathrm{staggered\,CS}}[a_\mu]. \tag{19}$$

Intrawire dual-fermion kinetic energy is described by

$$h_{\mathrm{wire}}^{\mathrm{dual}}[\tilde{\phi}, a_\mu] = \frac{-(-1)^y}{2\pi}a_{0,y}\partial_x\tilde{\phi}_y + \frac{u}{4\pi}(\partial_x\tilde{\phi}_y - a_{1,y})^2,$$

which has a similar form to Eq. (17) except that here the fermions minimally couple to the emergent gauge field a_μ. A Maxwell term for the gauge field appears in

$$\mathcal{S}_{\mathrm{MW}}[a_\mu] = \int_{t,x}\sum_y\left[\frac{1}{16\pi v}(\Delta a_{0,y})^2 - \frac{v}{16\pi}(\Delta a_{1,y})^2\right].$$

In the continuum limit we indeed recover an anisotropic Maxwell term $\mathcal{S}_{\mathrm{MW}} \sim \lambda_\mu(\epsilon_{\mu\nu\kappa}\partial_\nu a_\kappa)^2$ in the $a_2 = 0$ gauge with $\lambda_y = 0$. Note that $\lambda_y = 0$ is a property of the bare microscopic theory; finite λ_y is generated under renormalization with any nonzero interwire hopping amplitude w. Finally, $\mathcal{S}'_{\mathrm{staggered\,CS}} = \int_{t,x}\sum_y(-1)^y/(8\pi)\Delta a_{0,y}(a_{1,y+1} + a_{1,y})$ is the discrete analog of the staggered Chern-Simons coupling in Eq. (7), but for a_μ. This term similarly drops out in the continuum limit. It is worth emphasizing that the gauge field a_μ, which is introduced via exact Gaussian integrals over all real values, is noncompact—just as in the boson-vortex duality [39,40].

Refermionizing $\mathcal{S}_{\mathrm{dual}}$ and taking the continuum limit yields $\mathcal{S}_{\mathrm{QED}_3}$ of Eq. (2) in the special case $A_\mu = 0$. In the Supplemental Material [26], we perform the same analysis including the electromagnetic vector potential A_μ. We find that in the continuum limit the dual action above merely acquires an extra mutual Chern-Simons term $\mathcal{S}_{\rm CS}[a_\mu, A_\mu] = \int_{t,x,y}(1/4\pi)\epsilon_{\mu\nu\kappa}A_\mu\partial_\nu a_\kappa$, precisely as in Eq. (2). This completes our duality derivation—which we emphasize makes no assumption about the value of A_μ. In particular, by virtue of the mutual Chern-Simons term, uniform chemical potential for electrons $A_0 \neq 0$ translates to a nonzero orbital magnetization for dual fermions, while electrons in magnetic field B correspond to dual fermions at finite density $n_{\rm dual} = B/(4\pi)$. Our analysis thus immediately applies both to the TI surface and half-filled Landau level.

Duality for gapped phases.—As a first application we briefly discuss interaction-induced gapped phases descending from the Dirac cone and QED_3 theories at $B = 0$. References [17,18] argued phenomenologically that a Fu-Kane superconductor [41] of dual fermions corresponds to "T-Pfaffian" non-Abelian topological order [13,14] for electrons and vice versa. Our explicit duality mapping allows us to readily translate the dual-fermion interactions needed for the former (which are straightforward to obtain) into an electronic Hamiltonian for the T-Pfaffian (which is highly nontrivial).

We simply sketch the construction here; for details see [26]. First, within our wire formulation it proves convenient to enlarge the unit cell and relax the symmetries in Eq. (5) by enforcing only $\mathcal{T}^3$ and $\mathcal{C}^3$. One can then construct a coupled-wire Hamiltonian with three (electronic and dual) Dirac cones—two of which can be gapped without breaking symmetries. At low energies one thus again obtains the continuum theories (1) and (2) with symmetries acting according to Eqs. (6) and (14), but now with auxiliary gapped Dirac cones. Suppose that we specifically add interactions that gap the auxiliary dual Dirac cones through a pairing instability that spontaneously breaks dual-fermion number conservation. This condensate induces a proximity effect on the remaining massless dual Dirac cone and drives the dual fermions into a Fu-Kane superconductor. Quite remarkably, dualizing these interactions yields precisely the electron Hamiltonian found in Ref. [42] to generate the T-Pfaffian for the electron TI surface. This provides a strong consistency check on our analysis.

Properties of $N = 1$ QED_3.—Our explicit duality mapping also allows nontrivial predictions for QED_3, a naively strongly interacting theory, by relating operators to their free-electron counterparts. As an important example, Eq. (13) immediately gives

$$m\int_x \sum_y (\tilde{\psi}_y^\dagger \tilde{\psi}_{y+1} + \mathrm{H.c.}) = m\int_x \sum_y (\psi_y^\dagger \psi_{y+1} + \mathrm{H.c.}). \tag{20}$$

In the continuum limit the left and right sides respectively yield the dual-fermion mass term $m\tilde{\Psi}^\dagger\sigma^1\tilde{\Psi}$ and electron mass term $m\Psi^\dagger\sigma^1\Psi$, and hence the two are identified. It follows that the scaling dimension for the mass in QED_3 must be *precisely* 2. To appreciate this result, note that in QED_3 with N fermion flavors, a large-N treatment predicts a positive $O(1/N)$ correction to the free-fermion result arising from gauge-field-mediated interactions [43]. Our results imply that such corrections must exactly cancel when summed to all orders in $1/N$ in the $N = 1$ limit—even though the fermions in QED_3 strongly couple to the gauge field. In [26] we additionally explore analogs of Eq. (20) for fermion currents. Constraints from current conservation allow us to make exact statements about scaling dimensions in this case, providing further consistency checks on the duality.

Strictly speaking, the exact duality mapping between lattice models holds for $N = 1$ QED_3 defined with charge of order unity on the lattice scale. An alternative definition with infinitesimal charge on the lattice scale may, in principle, flow to a different low-energy fixed point that need not be critical, but given our consistency checks this seems unnatural. Within our regularization the existence of a gapless phase is unambiguous by virtue of the exact mapping to free Dirac electrons.

Duality for $N = 2$ QED_3.—It is interesting to ask whether similar mappings exist for larger-N QED_3. To address this question, and illustrate our method's generality, we run our formalism "in reverse" to determine the theory dual to $N = 2$ QED_3. The latter corresponds to two copies of the wire model (19) labeled by $\sigma = \pm$, each with its own species of dual fermions $\tilde{\psi}_\sigma \sim e^{i\tilde{\phi}_\sigma}$ but with the same dynamical gauge field a_μ. The intrawire Hamiltonian density expressed in terms of (copropagating) charge and neutral modes $\tilde{\phi}_c = (\tilde{\phi}_+ + \tilde{\phi}_-)/2$ and $\tilde{\phi}_n = (\tilde{\phi}_+ - \tilde{\phi}_-)/2$ is $h^{\rm dual}_{\rm wire}[\tilde{\phi}_c, a_\mu] + h_{\rm wire}[\tilde{\phi}_n]$. (Here, charge and neutral are with respect to the dynamical gauge field a_μ rather than the external vector potential A_μ.) Since only $\tilde{\phi}_c$ couples to a_μ we implement the duality by defining a chiral dual charge mode $\phi_{c,y} = \sum_{y'} D_{y,y'}\tilde{\phi}_{c,y'}$ that counterpropagates relative to the neutral mode $\tilde{\phi}_{n,y}$, which we leave intact. Integrating out a_μ then yields a local theory for ϕ_c with intrawire kinetic energy $h_{\rm wire}[\phi_c]$. Interwire hopping becomes [cf. Eq. (13)]

$$\sum_\sigma \tilde{\psi}^\dagger_{\sigma,y}\tilde{\psi}_{\sigma,y+1} + \mathrm{H.c.} \to \sum_\sigma b^\dagger_{\sigma,y} b_{\sigma,y+1} + \mathrm{H.c.}, \tag{21}$$

where $b^\dagger_{\sigma,y} \sim e^{-i\phi_{c,y} - \sigma i\tilde{\phi}_{n,y}}$ creates a dual boson of species σ. This suggests introducing nonchiral modes $\varphi_{\sigma,y} = (\phi_{c,y} + \sigma\tilde{\phi}_{n,y})$ that obey the intrawire action

$$\mathcal{S}_{\rm wire} = \int_{t,x}\sum_y \left[\frac{(-1)^y K_{\sigma\sigma'}\partial_x\varphi_{\sigma,y}\partial_t\varphi_{\sigma',y}}{4\pi} + \frac{v_\sigma}{4\pi}(\partial_x\varphi_{\sigma,y})^2\right] \tag{22}$$

with K matrix $K = \sigma^x$.

The network model described by Eqs. (21) and (22) was introduced in Ref. [33] to describe the surface of time-reversal-invariant bosonic TIs with two separately conserved $U(1)$ symmetries. We now see explicitly that $N = 2$ QED_3 provides a dual description of the bosonic TI surface [44].

This duality was proposed based on bulk arguments in Refs. [45,46]. Unlike the duality for $N = 1$ QED$_3$, however, the $N = 2$ analog does not map onto a simple free theory. Thus, one cannot readily infer operator scaling dimensions in this gauge theory by mapping between the two. Inspired by its utility in the fermion case, we nevertheless expect that the duality will be conceptually useful, e.g., for accessing novel surface phases of bosonic TIs.

Conclusions.—We derived the equivalence of a free Dirac cone and $N = 1$ QED$_3$ on the level of the path integral. We used this mapping to determine the scaling dimension of the fermion mass in $N = 1$ QED$_3$, and illustrated with the example of the T-Pfaffian how this formulation can be used to easily obtain electron Hamiltonians for topologically ordered phases. By running the same mapping in reverse we found that $N = 2$ QED$_3$ is dual to a critical theory relevant for bosonic TI surfaces. We expect that generalizations of our approach will provide insights into other exotic phases with emergent gauge fields, such as gapless quantum spin liquids.

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